

1 **THE NETWORK HHD: QUANTIFYING CYCLIC COMPETITION IN**
2 **TRAIT-PERFORMANCE MODELS OF TOURNAMENTS ***

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4 **Abstract.** Competitive tournaments appear in sports, politics, population ecology, and animal
5 behavior. All of these fields have developed methods for rating competitors and ranking them accord-
6 ingly. A tournament is intransitive if it is not consistent with any ranking. Intransitive tournaments
7 contain rock-paper-scissor type cycles. The discrete Helmholtz-Hodge decomposition (HHD) is well
8 adapted to describing intransitive tournaments. It separates a tournament into perfectly transitive
9 and perfectly cyclic components, where the perfectly transitive component is associated with a set of
10 ratings. The size of the cyclic component can be used as a measure of intransitivity. Here we show
11 that the HHD arises naturally from two classes of tournaments with simple statistical interpretations.
12 We then discuss six different sets of assumptions that define equivalent decompositions. This analysis
13 motivates the choice to use the HHD among other existing methods. Success in competition is often
14 mediated by the traits of the competitors. A trait-performance model assumes that the probability
15 that one competitor beats another is a function of their traits. We show that, if the traits of each
16 competitor are drawn independently and identically from a trait distribution then the expected de-
17 gree of intransitivity in the network can be computed explicitly. We show that increasing the number
18 of pairs of competitors who could compete promotes cyclic competition, and that correlation in the
19 performance of A against B with the performance of A against C promotes transitive competition.
20 The expected size of cyclic competition can thus be understood by analyzing this correlation.

21 **Key words.** Cyclic competition, intransitivity measures, least squares rating, Helmholtz-Hodge
22 decomposition, trait-performance models

23 **AMS subject classifications.** 05C50, 05C20, 05C21

24 **1. Introduction: Tournaments, Ranking, and Intransitivity.** A tourna-
25 ment consists of a group of competitors who compete in pairwise events (head-to-head
26 matches). Tournaments are important across disciplines, from ecology and animal be-
27 havior [43, 63], to psychology and sports [6, 35]. Rating and ranking, that is, assigning
28 a measure of quality to the competitors and listing them in order from best to worst,
29 is important in each of these areas. In sports, ranking and rating teams and players
30 is a topic of broad popular interest. In biology, ratings are widely used to evaluate
31 the quality of competitors in social hierarchies. High standing in a competitive hier-
32 archy may be closely related to fitness, as it is often associated with priority access to
33 resources [17, 38, 39, 69], territory maintenance [64], and higher reproductive output
34 [54, 75]. Ranking is especially important in politics, as many electoral systems deter-
35 mine a winner by aggregating votes into a partial ranking of the candidates. Ratings
36 and rankings are often sought since they simplify the description of a tournament by
37 assigning each competitor a single number that purports to measure how good they
38 are.

39 Not every tournament allows for a consistent ranking of competitors. As a moti-
40 vating example, consider the 2019–2020 National Basketball Association (NBA) sea-
41 son, which was cut short by the COVID-19 pandemic. Imagine two fans arguing
42 whether the Cleveland Cavaliers (CLE) or Sacramento Kings (SAC) were the better
43 team. The two teams did not play in 2019–2020 due to the abbreviated season, so

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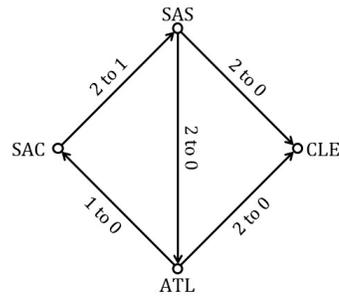


FIG. 1. A network representing the observed outcomes of games between the Cleveland Cavaliers (CLE), Sacramento Kings (SAC), Atlanta Hawks (ATL), and San Antonio Spurs (SAS) in the 2019–2020 regular season. Arrows point from the team which lost the majority of the games to the team which won the majority. Labels next to the arrows provide the game outcomes.

44 they cannot be compared directly. The Cleveland fan points out that CLE beat the
 45 San Antonio Spurs (SAS) 2 out of 2 games, and SAS beat SAC 2 out of 3 games, so
 46 surely CLE was better than SAC. The SAC fan counters that transitive predictions
 47 of this kind are not always valid. For example, the Atlanta Hawks (ATL) beat SAS
 48 2 out of 2 games, and SAS beat SAC 2 out of 3 games, yet SAC still beat ATL in
 49 the game they played. Figure 1 illustrates these outcomes as a graph. Notably, the
 50 graph contains a mixture of triangles which do and do not allow consistent rankings.
 51 A believer in ranking could point to the triangle involving CLE, SAS and ATL as
 52 evidence that NBA teams can be consistently ranked, while a skeptic might point to
 53 the triangle involving SAS, ATL, and SAC.

54 The observation that not all tournaments admit consistent rankings motivates
 55 classification into transitive and intransitive tournaments. A tournament is *transitive*
 56 if knowing that A usually beats B , and B usually beats C , is enough to conclude that
 57 A usually beats C . Transitive tournaments are consistent with a global ranking of all
 58 the competitors. An *intransitive* tournament is a tournament that is not consistent
 59 with any global ranking. Intransitive tournaments must contain at least one cycle
 60 where the transitive assumption fails. Figure 2 illustrates examples of transitive and
 61 intransitive tournaments.

62 Intransitive tournaments appear in practically every discipline where tournaments
 63 are studied [10, 23, 52, 57, 59], and are the norm rather than the exception when using
 64 real data [32, 35, 36, 43, 63, 66, 68]. Intransitivity may arise due to uncertainty in
 65 observed data [35, 68], randomness in event outcomes, or may be intrinsic, as in the
 66 game of rock-paper-scissors.

67 Intransitivity is important for two reasons. First, intransitivity presents a chal-
 68 lenge when ranking since no ranking is consistent with the tournament. For example,
 69 Condorcet’s paradox is a voting paradox in cyclic community preferences prevent any
 70 fair ranking of candidates, and thus, any choice of winner [23].¹ Second, when intran-
 71 sitivity is intrinsic, then the tournament contains cyclic structure, as in rock-paper-
 72 scissors. Cyclic structures can radically alter optimal strategies [10] and long term
 73 dynamics [52, 59, 58, 60, 61]. For example, in ecology it is widely hypothesized that
 74 intransitive competition between species promotes biodiversity since no species domi-

¹Suppose there are three candidates in an election and three voters. Suppose that the first voter prefers A to B to C , the second B to C to A , and the third C to A to B . Then A would beat B , B would beat C , and C would beat A in pairwise head-to-head elections.

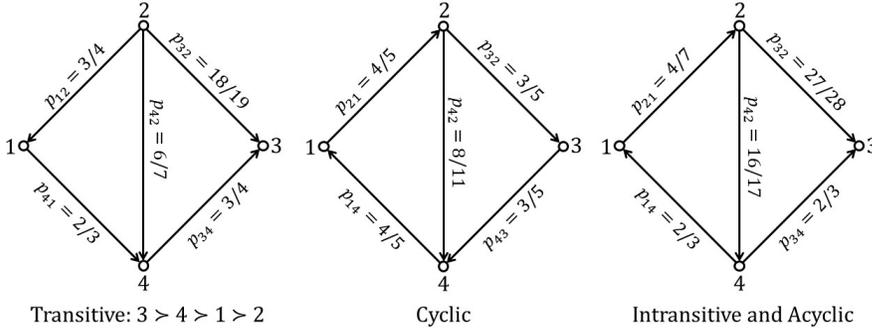


FIG. 2. Three example networks representing different classes of tournaments. The first is transitive since the win probabilities are consistent with the ranking $3 > 4 > 1 > 2$. The second is both intransitive, and, more strongly, is cyclic (see Section 3.1.2 for definitions). The third is neither transitive nor cyclic, and represents a generic tournament, with the same pattern of expected winners and losers as in Figure 1.

75 nates. This hypothesis is based on extensive theoretical work [43, 52, 59, 58, 60, 61, 80]
 76 and limited case-studies [30, 36, 46, 47, 66].

77 The importance of intransitivity in real natural communities is controversial [25,
 78 70, 77], in part because there are few robust metrics for measuring intransitivity
 79 from incomplete and noisy data. Uncertainty in data can easily be conflated with
 80 observed intransitivity, and common sampling methods for filling in missing data can
 81 overestimate intransitivity [63]. Thus there is a need for ranking and rating methods
 82 that are robust to intransitivity and measures of intransitivity that can handle noisy
 83 and incomplete data.

84 Jiang and Lim introduced the discrete Helmholtz-Hodge Decomposition (HHD)
 85 as a general method for ranking objects from incomplete and imbalanced data [32,
 86 50]. The decomposition is a network theoretic tool that we adapt to the study of
 87 competitive tournaments. The HHD accomplishes three fundamental tasks. First, it
 88 assigns a rating to each competitor. Competitors can be ranked accordingly. Second,
 89 it produces a measure of intransitivity that quantifies how far an observed network
 90 is from the nearest perfectly transitive network. Third, it represents the observed
 91 network as the direct sum of a perfectly transitive and a perfectly cyclic network.
 92 This decomposition provides an elegant characterization of intransitivities present in
 93 data, and can reveal underlying cyclic tendencies (c.f. [10]).

94 When compared to existing methods, the discrete HHD has a number of advan-
 95 tages. It is more general than some classical methods since it applies to arbitrary
 96 network topologies and can accommodate imbalanced data [32]. It is also more infor-
 97 mative because it provides a clear description of both underlying transitive and cyclic
 98 structures. Most ranking methods and intransitivity measures focus on the transi-
 99 tive component while the HHD puts the transitive and cyclic components on equal
 100 footing. Finally, it remains efficiently computable even for large, incomplete networks
 101 [32]. In contrast, Slater's index [68] requires solving an NP hard optimization problem
 102 [11, 18], and Kendall's index [35] requires a complete network.

103 This paper aims to answer two fundamental questions:

- 104 1. Why use the HHD to study competition when other methods exist?
- 105 2. Having chosen to use the HHD, what do we expect when competitive perfor-
 106 mance is determined by individuals' traits?

107 Answering the first question is important since there are many possible meth-
 108 ods to choose from, so the choice of method should be made in a principled way.
 109 Answering the second question is important since it builds a conceptual bridge from
 110 the competitors and competitive event to the overall tournament structure. As in
 111 Landau [44], we seek to understand how the underlying distribution of traits among
 112 competitors, and the relationship between traits and success, influence the overall
 113 tournament.

114 The latter question is important across disciplines. In some biological settings,
 115 success in competition is determined by individual traits, driving selection [76]. For
 116 example, competition for social dominance among male elephant seals depends on
 117 their body mass [26] and competition among male dwarf Cape chameleons depends
 118 on coloration, head size, and body length [76]. Success in these competition events is
 119 correlated with reproductive success, suggesting that heritable traits which improve
 120 a male's chances of success are strongly selected for [26]. In sports, the relationship
 121 between the traits of a player or team and their success is an area of active interest - for
 122 athletes, owners, fans, and researchers alike. The rise of sabermetrics, the statistical
 123 study of baseball, is a popular example [49, 78].

124 This paper answers questions 1 and 2 as follows:

- 125 **1.** The HHD arises naturally from the study of ranking and intransitivity. To
 126 illustrate this point, we provide a different derivation of the HHD than [32] or
 127 [50]. Instead of imposing the decomposition ad hoc, we propose two special
 128 classes of tournaments with clear statistical motivation. We then show that
 129 any tournament can be uniquely decomposed into a combination of tourna-
 130 ments from these classes. This decomposition is the HHD (see [Theorem 3.5](#)).
 131 Next we illustrate that the HHD can be reached by six different approaches
 132 ([Corollary 3.6](#)), and is thus robust to varying motivations.
- 133 **2.** We show that, under simple assumptions on the distribution of traits, the
 134 expected sizes of the components of the decomposition can be computed ex-
 135 plicitly from the number of competitors, number of pairs who could compete,
 136 and the correlation in the performance of A against B with A against C . This
 137 correlation is shown to equal the uncertainty in the expected performance of
 138 a competitor, linking a decomposition of uncertainty in performance to tour-
 139 nament structure (see [Theorem 4.1](#) and [Corollary 4.2](#)).

140 The answers to the second question prove, under minimal assumptions, a series
 141 of intuitive statements about transitive/cyclic competition that appear, as heuristics,
 142 across the literature. These include:

- 143 (a) The more predictable the performance of A against a randomly drawn competitor
 144 (i.e., the less the performance of A depends on their opponent) the more transitive
 145 the tournament.
- 146 (b) The more correlated the performance of A against B with the performance of A
 147 against C , the more transitive the tournament.
- 148 (c) The more pairs of competitors who could compete, the more cyclic the tournament
 149 is, on average.
- 150 (d) Filling in missing data by random sampling overestimates intransitivity.

151 Statements a, b, and c also hold in reverse. Decreasing a quantity that promotes
 152 transitivity promotes cyclic competition

153 The paper is structured as follows. In [Section 2](#) we provide some necessary back-
 154 ground. Next, in [Section 3](#), we derive the HHD in the context of tournaments and de-
 155 velop the associated ratings and intransitivity measure. Our derivation complements
 156 the cohomological approach used by [50], as it is specially adapted to tournaments,

157 and only requires linear algebra and classical graph theory. In Section 4 we show
 158 how assumptions about the statistics underlying competition promote or suppress in-
 159 transitivity. We focus on trait-performance models in which performance is assumed
 160 to be a function of traits sampled from a trait distribution. While win probabilities
 161 are not always determined by traits, exploring trait determined performance affords a
 162 more realistic and richer perspective than standard null models (c.f. [16]), and demon-
 163 strates generic relationships. In particular, we present a theorem (4.1) which allows
 164 the expected size of the intransitivity measure to be computed directly from the num-
 165 ber of competitors, edges in the network, and correlation in the performance of A
 166 against B with A against C . This result is extended by a corollary (4.2) which shows
 167 that the correlation in performance is related to a decomposition in the uncertainty
 168 of the performance of A against B . These results lead to a deeper conceptual un-
 169 derstanding of how cyclic structure can arise from uncertainty in performance, and
 170 can be suppressed by correlation in performance. We conclude by generalizing these
 171 observations to scenarios where the trait-performance assumptions do not hold.

172 **2. Mathematical Framework.** Consider an ensemble of V competitors. As-
 173 sume that each competition event involves exactly two competitors, and never results
 174 in a tie. This standard assumption [35, 43] can be weakened to allow for ties. We will
 175 refer to competition of this kind as a tournament.²

176 The probability of any sequence of event outcomes in a tournament is determined
 177 by the probabilities that competitors beat each other. If the event outcomes are
 178 independent, then for each possible pairing of competitors there is an unambiguous
 179 probability one beats the other. Let p_{AB} denote the probability competitor A beats B .
 180 The shorthand $A \succ B$ denotes the case when A is expected to beat B ($p_{AB} > 1/2$). In
 181 principle, the win probabilities could change in time, and could depend on the history
 182 of the process (c.f. [24]). We focus on tournaments with unchanging win probabilities
 183 to avoid modeling additional temporal dynamics. Then a fixed set of win probabilities
 184 p determine the probability of any sequence of events. Thus the tournament dynamics
 185 are realizations of a random process, with probabilities controlled by p and the event
 186 order. The event order, i.e. the schedule, could be fixed or random. As in other studies
 187 of transitivity, we focus on the structure of the win probabilities p , not the schedule
 188 or tournament dynamics, since the win probabilities p determine the distribution of
 189 possible tournament outcomes, and whether competition is transitive or intransitive.

190 The win probabilities may be conveniently represented using a competitive net-
 191 work, $\mathcal{G}_{\rightleftharpoons} = (\mathcal{V}, \mathcal{E}, p)$. Assign each competitor a node from the vertex set \mathcal{V} . Then
 192 $V = |\mathcal{V}|$. Introduce a pair of directed edges between each pair of competitors who
 193 could compete. The edge from B to A is assigned the weight p_{AB} . We assume that
 194 the tournament is finite, *connected* and *reversible*. That is, there are finitely many
 195 competitors, for any pair of competitors $A B$ there is a path from A to B and from B
 196 to A through $\mathcal{G}_{\rightleftharpoons}$ with probability greater than zero, and that $p_{AB} \neq 0$ or 1 for any
 197 pair A, B who could compete.

198 Sometimes it is preferable to simplify the competition network by rounding all
 199 weights less than $1/2$ to 0, and all weights greater than $1/2$ to 1. This can be conve-
 200 niently represented as an unweighted graph $\mathcal{G}_{\rightarrow}$ which contains all directed edges from
 201 $\mathcal{G}_{\rightleftharpoons}$ with weights greater than a half, and an undirected edge between all pairs with
 202 $p_{AB} = 1/2$. The edges in this graph point from expected losers to expected winners.
 203 Most intransitivity measures focus on this graph (see [35], [44], [68]).

²This is distinct from a *complete* tournament in which it must be possible for all pairs to compete.

204 A *ranking* is an ordered list of competitors from best to worst, specified by a
 205 rank function R which returns the rank of each competitor. Note that this is distinct
 206 from a *rating*, r , which is a function that returns a real number for each competitor
 207 [45]. Rankings are often generated by first rating each competitor, then listing them
 208 in decreasing order. For example, given competitors A, B, C with ratings $r_A = 10$,
 209 $r_B = 20$, $r_C = 0$ the corresponding ranking would be $R_A = 2$, $R_B = 1$, $R_C = 3$ and
 210 the competitors would be listed $B \succ A \succ C$. Ratings provide an intuitive description
 211 of competition in which some innate competitive ability determines performance.

212 Ranking methods are diverse, and well studied. Famous examples include the
 213 page-rank method used by Google to sort search results [9], the Massey and Col-
 214 ley methods used by the NCAA to rank basketball and football teams [45], and the
 215 Elo rating/ranking widely used by chess federations [24, 71]. The rating system pro-
 216 duced by the HHD is a kind of log-least squares rating as is frequently used in paired
 217 comparison [6, 41, 42]. Examples of least squares rating systems are included in
 218 [14, 34, 45, 51, 72, 73].

219 A competitive network $\mathcal{G}_{\rightleftharpoons}$ is consistent with a ranking R if $A \succ B$ whenever
 220 $R(A) < R(B)$. If a competitive network is consistent with a ranking then this ranking
 221 is unique and the network is *transitive*. Transitive networks satisfy the intuitive
 222 property that if we consider some sequence of competitors with $A \succ B \succ C \succ D$
 223 then $A \succ D$. That is, $\mathcal{G}_{\rightarrow}$ contains no cycles, and all the edges in $\mathcal{G}_{\rightarrow}$ point from
 224 competitors with worse ranks to competitors with better ranks.

225 If $\mathcal{G}_{\rightarrow}$ contains a cycle, then there exists a sequence of competitors such that $A \succ$
 226 $B \succ C \succ \dots \succ A$, and the tournament is *intransitive*. If a network is intransitive then
 227 it is not consistent with any ranking [57]. Speaking broadly, measures of intransitivity
 228 either count the number of intransitive triangles present in $\mathcal{G}_{\rightarrow}$ [35], or measure how
 229 far $\mathcal{G}_{\rightarrow}$ is from a nearby transitive network [68]. The Kendall measure [35] counts the
 230 number of intransitive triangles in $\mathcal{G}_{\rightarrow}$. This can be done efficiently, however prioritizes
 231 triangles over larger loops and does not weight edges equally [2, 68]. The Slater
 232 measure of intransitivity is the minimum number of edge directions that need to be
 233 reversed in order to transform $\mathcal{G}_{\rightarrow}$ into a transitive network [68]. While conceptually
 234 preferable [32], finding the closest transitive network is an NP hard problem [3], [19],
 235 [27], [32]. Despite some fast heuristics [18], complexity concerns limit the application
 236 of the Slater measure to small networks. The intransitivity measure associated with
 237 the HHD is conceptually analogous to the Slater measure, but can be computed
 238 efficiently even for very large networks. Note that transitivity and intransitivity are
 239 defined relative to the *sign* of $(p_{AB} - 1/2)$, rather than the exact value p_{AB} . In
 240 contrast, the intransitivity measure associated with the HHD is continuous in the win
 241 probabilities, so uses all the information available in $\mathcal{G}_{\rightleftharpoons}$.

242 **3. The Network HHD.** The Network Helmholtz-Hodge Decomposition (HHD)
 243 can be derived by defining two special classes of tournaments. These parallel the two
 244 classes of games defined in [10].

245 3.1. Arbitrage Free and Favorite Free Tournaments.

246 **3.1.1. Arbitrage Free Tournaments (Perfectly Transitive).** A currency
 247 market is said to be *arbitrage free* if it is impossible to make money by exchanging
 248 currencies cyclically [32]. By analogy, we define an *arbitrage free tournament* to be a
 249 tournament for which it is impossible to expect to make money by betting on cyclic
 250 sequences of events. Specifically, a tournament is arbitrage free if, for any cyclic
 251 sequence of competitors $\mathcal{C} = \{i_1, i_2, \dots, i_{|\mathcal{C}|}, i_{|\mathcal{C}|+1} = i_1\}$, a sequence of wins where i_j

252 loses to i_{j+1} is equally likely as a sequence of wins where i_j beats i_{j+1} for all j . Here
 253 $|\mathcal{C}|$ denotes the number of competitors in the cycle.

254 **Cycle Condition:** A tournament is arbitrage free if and only if, for every cycle
 255 $\mathcal{C} = \{i_1, i_2, \dots, i_{|\mathcal{C}|}, i_{|\mathcal{C}|+1} = i_1\}$, the win probabilities satisfy:

$$256 \quad (3.1) \quad p_{i_1 i_2} p_{i_2 i_3} \dots p_{i_{|\mathcal{C}|} i_1} = p_{i_1 i_{|\mathcal{C}|}} \dots p_{i_3 i_2} p_{i_2 i_1}.$$

257 The cycle condition can be simplified by dividing the right hand side across to
 258 the left hand side and taking a logarithm. Then:

$$259 \quad (3.2) \quad \sum_{j=1}^{|\mathcal{C}|} f_{i_j i_{j+1}} = 0$$

260 where the f_{ij} is the log-odds that competitor i beats competitor j :

$$261 \quad (3.3) \quad f_{ij} = \text{logit}(p_{ij}) \equiv \log\left(\frac{p_{ij}}{1 - p_{ij}}\right).$$

262 The cycle condition is satisfied if and only if the sum of f around any cycle is
 263 zero. The log-odds, f , are an example of an *edge flow*: an alternating function,
 264 $f_{ij} = -f_{ji}$, on the edges [32]. Note that $\text{logit}(x) = \log(x/(1-x))$ is the inverse of
 265 $\text{logistic}(y) = 1/(1 + \exp(-y))$, so no information is lost in moving to f from p .

266 The sum of f around a cycle is an example of a path sum. A *path sum* against
 267 an edge flow is the discrete analog of a path integral against a vector field. Given a
 268 sequence of competitors $\mathcal{P} = \{i_1, i_2, \dots, i_{|\mathcal{C}|}\}$ the path sum against f over the path \mathcal{P}
 269 is $\sum_{j=1}^{|\mathcal{C}|-1} f_{i_{j+1} i_j}$. The cycle condition requires that path sums over cycles equal zero.

270 If path integrals around closed loops equal zero, then the value of path integrals
 271 depend only on the endpoints of the path, are otherwise path independent, and the
 272 vector field is the gradient of potential. These properties also hold for networks.

273 **LEMMA 3.1 (Arbitrage Free).** *A tournament is arbitrage free if and only if there*
 274 *exists a unique set of ratings r , with average rating equal to zero, such that the win*
 275 *probabilities satisfy $p_{ij} = \text{logistic}(r_i - r_j)$. Moreover if a tournament is arbitrage free*
 276 *then it is transitive.*

277 If there exist a set of ratings such that $p_{ij} = \text{logistic}(r_i - r_j)$ then $f_{ij} = r_i - r_j$
 278 so path sums over f are telescoping, and thus cancel around loops. Then the cycle
 279 condition holds automatically. The rest of Lemma 3.1 can be proved using a simple
 280 spanning tree construction illustrated in of Figure 3 (panel a). We sketch the proof
 281 here; the supplement provides further details.

282 If a network is arbitrage free then the cycle condition requires that the path sum
 283 of f around any loop is zero. It follows that path sums over f are path independent.
 284 Our goal is to find a rating r such that the difference in r on each edge produces the
 285 edge flow f . We recover r by picking a spanning tree³, and assigning it an arbitrary
 286 root, A . Uncentered ratings u are computed by setting u_i equal to the path sum from
 287 A to node i along the paths in the tree. Then the ratings r are set equal to u_i minus
 288 the average value of u . Path independence guarantees that the choice of tree does
 289 not influence u , and centering the ratings eliminates any dependence on the choice of

³A spanning tree is a subgraph of the network that contains no loops, includes all competitors, and is connected.

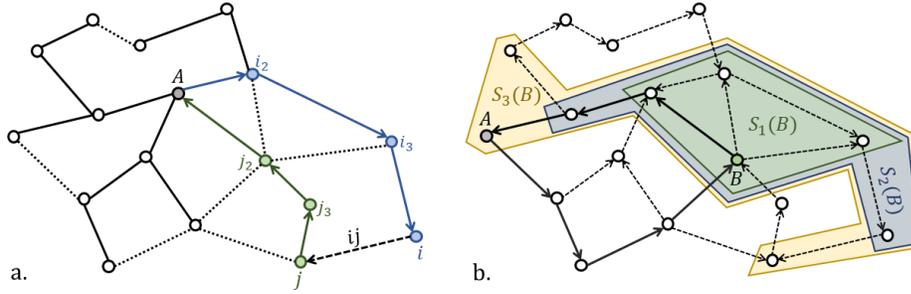


FIG. 3. Panel a. The spanning tree construction for recovering the ratings for an arbitrage-free tournament. The tree is shown with solid lines, and the chords with dotted lines. The root of the tree, A , is marked in grey. Two vertices, i and j connected by a chord ij , are shown in blue and green respectively. The sequence of nodes leading from A to i and j are labelled. If the ratings, r , are constructed by evaluating path sums over the tree, then the path sum from j to A is $r_j - r_A$, and the path sum from A to i is $r_A - r_i$. Then, by the cycle condition, the sum around the loop marked with arrows is zero, hence $f_{ij} = r_i - r_j$. Panel b. A favorite free tournament must be a cyclic tournament. The arrows represent the direction of competition. If the network is favorite free, then whenever there is an edge pointing into a set there must be an edge pointing out of it. A path from A to B is shown in black. Then the sets $S_1(B)$, $S_2(B)$, $S_3(B)$ are shown as shaded polygons. These contain all competitors distance 1, 2, and 3 (respectively) from B . These sets continue to expand until they include A , hence there is a path from B to A .

290 A. Then, by construction, $r_i - r_j = f_{ij}$ on all edges in the tree. The cycle condition
 291 guarantees that $r_i - r_j = f_{ij}$ on all edges not in the tree. Since f are the log-odds,
 292 $p_{ij} = \text{logistic}(r_i - r_j)$. Transitivity follows automatically since p must be consistent
 293 with the ranking induced by r .

294 **Lemma 3.1** shows that arbitrage free tournaments are the only tournaments which
 295 match the logistic rating model $p_{ij} = \text{logistic}(r_i - r_j)$ used for Elo rating [1, 29, 45].⁴

296 Arbitrage free tournaments are also the only tournaments that match the Bradley-
 297 Terry model:⁵ $p_{ij} = q_i / (q_i + q_j)$ where $q_i \geq 0$ are the Bradley-Terry ratings [8, 7].
 298 If a network is arbitrage free, then setting $q_i = \exp(r_i)$ recovers the Bradley-Terry
 299 model. If the tournament satisfies the Bradley-Terry model, then setting $r_i = \log(q_i)$
 300 produces a rating which satisfies $p_{ij} = \text{logistic}(r_i - r_j)$, so the network must be
 301 arbitrage free.

302 Since arbitrage free networks are a special class of transitive networks, we will
 303 refer to them as “perfectly” transitive. Note that a perfectly transitive network must
 304 satisfy the cycle condition, which is a requirement on the values of p rather than the
 305 sign of $(p - 1/2)$. Hence, while all perfectly transitive networks are transitive, not all
 306 transitive networks are perfectly transitive. For example, if $p_{AB} = 0.99$, $p_{BC} = 0.99$,
 307 and $p_{AC} = 0.51$ then the tournament is transitive, even though p_{AC} is much smaller
 308 than might be expected. This example is not perfectly transitive since it does not
 309 satisfy the cycle condition. The leftmost network in Figure 2 is perfectly transitive.

⁴The Elo rating system was originally proposed to rate chess players, but is also used to rank Sumo wrestlers [71], English league football teams [29] and international football teams. In the latter example the Elo method was the most predictive out of all methods tested [48]. The Women’s World Cup uses a variant on the Elo method [48].

⁵The Bradley-Terry model is widely used in pairwise comparison and to rank competitors in tournaments. Examples include professional tennis [40, 53], Cape dwarf chameleons [76] and northern elephant seals [26]. Bradley-Terry models accounting for surface type, and discounting old games, have been shown to be effective in predicting the outcome of ATP tennis tournaments [53].

310 **3.1.2. Favorite Free Tournaments (Perfectly Cyclic).** In contrast, we de-
 311 fine a *favorite free tournament* to be a tournament for which it is impossible to make
 312 money on average by betting on a favorite competitor over their neighbors. Specifi-
 313 cally, A is equally likely to beat all of their neighbors, as to lose to them. Let $\mathcal{N}(i)$
 314 denote the neighborhood of i , the set of all competitors who could compete with i .
 315 Then the win probabilities must satisfy a neighborhood condition.

316 **Neighborhood Condition:** A tournament is favorite free if and only if, for
 317 every competitor i with neighborhood $\mathcal{N}(i)$, the win probabilities satisfy:

$$318 \quad (3.4) \quad \prod_{j \in \mathcal{N}(i)} p_{ij} = \prod_{j \in \mathcal{N}(i)} p_{ji}.$$

319 Like the cycle condition, the neighborhood condition can be written directly as a
 320 condition on the log-odds edge flow f defined in equation (3.3). A tournament satisfies
 321 the neighborhood condition if and only if the sum of f_{ij} over the neighborhood of i
 322 is zero for all competitors i :

$$323 \quad (3.5) \quad \sum_{j \in \mathcal{N}(i)} f_{ij} = 0.$$

324 If the neighborhood condition is satisfied then it can be extended to all sets of
 325 competitors. Let S be a set of competitors and let $\mathcal{N}(S)$ be the set of all competitors
 326 not in S who neighbor S . Then the neighborhood condition implies:

$$327 \quad (3.6) \quad \sum_{j \in \mathcal{N}(S), i \in S} f_{ij} = 0.$$

328 This identity follows from the discrete analog to the divergence theorem: the sum of
 329 f over the neighborhood of S equals the sum of f over the neighborhood of every
 330 competitor in S .⁶ Then $\sum_{j \in \mathcal{N}(S), i \in S} f_{ij} = \sum_{i \in S} \sum_{j \in \mathcal{N}(i)} f_{ij} = \sum_{i \in S} 0 = 0$.

331 The cycle condition defined a special subset of transitive tournaments. The neigh-
 332 borhood condition also defines a special class that is a subset of a larger class - the
 333 class of cyclic tournaments. A *cyclic tournament* is a tournament such that, if there
 334 is a path from A to B in $\mathcal{G}_{\rightarrow}$, then there is a path back from B to A in $\mathcal{G}_{\rightarrow}$.

335 **LEMMA 3.2** (favorite free). *A favorite free tournament is cyclic, and is never*
 336 *transitive unless $p_{ij} = 1/2$ for all connected i, j .*

337 Like Lemma 3.1, Lemma 3.2 can be proved with a simple construction. The proof
 338 is sketched here and illustrated in Figure 3 (panel b). See supplement for details.

339 If there is a path from A to B in $\mathcal{G}_{\rightarrow}$ then we need to construct a path back
 340 to A from B . To this end, we define a nested sequence of sets where $S_d(B)$ is all
 341 vertices within distance from d of B in $\mathcal{G}_{\rightarrow}$. The neighborhood condition extends to
 342 sets of vertices, so if there is an edge into a set S in $\mathcal{G}_{\rightarrow}$ then there must also be
 343 an edge leaving S . It follows that, if A is not in $S_d(B)$, then $S_{d+1}(B) \neq S_d(B)$, so
 344 we can keep expanding the sequence of nested sets. If the network is finite then the
 345 sets cannot expand forever without eventually including A . To finish, a favorite free
 346 tournament cannot be transitive unless it is neutral, $p_{ij} = 1/2$ for all i, j , since only
 347 neutral tournaments are simultaneously transitive and cyclic.⁷

⁶If i and j are both in S then the sum over the neighborhood of i contributes f_{ij} , and the sum over the neighborhood of j contributes $f_{ji} = -f_{ij}$. Therefore the edge flow on any edge connecting a pair of nodes in S cancels in the sum.

⁷Note that a neutral tournament is considered transitive since it can be consistently ranked - all competitors should be ranked the same.

348 So, just as the cycle condition (no tendency to cycle) implied transitivity, the
 349 neighborhood condition, (no favorites) implies that the network is cyclic. Whether a
 350 tournament is cyclic or not depends on the sign of $(p_{ij} - 1/2)$, while the neighbor-
 351 hood condition is a condition on the values of p_{ij} . This motivates the definition: a
 352 tournament is *perfectly cyclic* if and only if it is favorite free. As before, all perfectly
 353 cyclic tournaments are cyclic, but not all cyclic tournaments are perfectly cyclic. The
 354 middle network in [Figure 2](#) is perfectly cyclic.

355 Note that, unlike perfectly transitive tournaments where f is determined by a set
 356 of ratings r , we are not currently equipped to relate the edge flow of a favorite free
 357 tournament to a lower dimensional representation. In [Subsection 3.2.2](#) we will show
 358 that a favorite free tournament has edge flows f which can always be represented as
 359 a sum of cyclic intensities (or vorticities) on a set of loops. This result will parallel
 360 the conclusions of [Lemma 3.1](#).

361 **3.2. The Discrete HHD.** Given these two classes of tournaments it is natural
 362 to ask: can a generic tournament be decomposed into a perfectly transitive (arbitrage
 363 free) part and a perfectly cyclic (favorite free) part? We answer in the affirmative.
 364 This is the Helmholtz-Hodge decomposition.

365 **3.2.1. Operators.** In order to define the decomposition succinctly, it is helpful
 366 to have a pair of operators analogous to the gradient and curl operators in the contin-
 367 uum. We simplify the topological presentation in [\[32\]](#) by expressing the decomposition
 368 entirely through linear algebra. For a cohomological discussion see [\[50\]](#).

369 First, define the edge space \mathbb{R}^E , where E is the number of pairs i, j who could
 370 compete. Index each pair so that edge k points from competitor $j(k)$ to competitor
 371 $i(k)$. Note that this requires assigning each edge an arbitrary start and endpoint.
 372 Positive f indicates that the competitor at the end is expected to beat the competitor
 373 at the start, and negative f indicates the reverse. This is simply a sign convention.

374 Let the *discrete gradient* operator G be the $E \times V$ matrix which maps from \mathbb{R}^V
 375 to \mathbb{R}^E by setting:

$$376 \quad (3.7) \quad [Gu]_k = u_{i(k)} - u_{j(k)}.$$

377 Then $g_{kh} = 1$ if $h = i(k)$, equals -1 if $h = j(k)$, and is zero otherwise. The matrix G
 378 is sometimes called the edge incidence matrix since it records the start and end point
 379 of each edge.

380 Notice that if r is a rating function on the nodes, then attempting to find r such
 381 that $r_i - r_j = f_{ij}$ is equivalent to looking for r such that $Gr = f$. Since any arbitrage
 382 free tournament admits a unique rating r satisfying $Gr = f$, the space of perfectly
 383 transitive competitive networks is equivalent to the space of competitive networks
 384 with edge flow f in the range of the gradient.⁸

385 The gradient transpose, G^T is the discrete divergence operator. The divergence
 386 maps from the space of edges to the space of nodes (competitors) such that:

$$387 \quad (3.8) \quad [G^T f]_i = \sum_{j \in \mathcal{N}(i)} f_{ij}.$$

388 The neighborhood condition [\(3.5\)](#) is equivalent to requiring that $G^T f = 0$. That
 389 is, the space of favorite free tournaments is equivalent to the space of tournaments
 390 with edge flow f in the null space of the divergence.

⁸Assuming that the competitive network is connected, the gradient has a one-dimensional null-

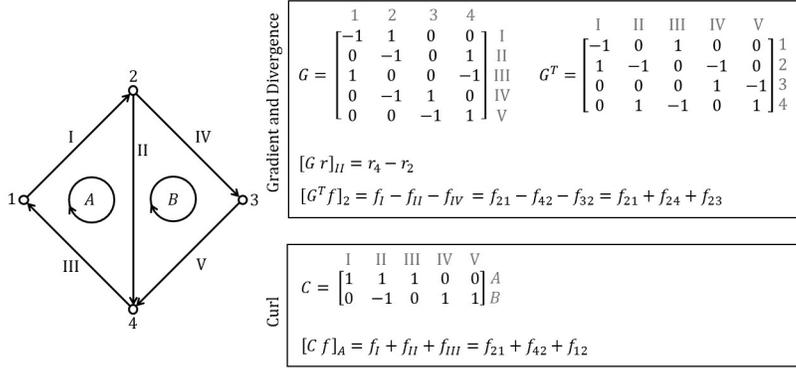


FIG. 4. The gradient, divergence, and curl for the example networks in Figure 2. A spanning tree for networks of this form could consist of edges I, II, and IV. Then the edges III and V are the chords, and the associated loops are the triangles labelled A and B.

391 In order to build a parallel description for perfectly cyclic tournaments, we need
 392 a space of loops. First define the sum of two cycles C_1, C_2 to be all edges included
 393 in either C_1 or C_2 but not both. Equipped with this addition operation, the space of
 394 cycles is a vector space, which can be represented with a cycle basis. A *cycle basis* is
 395 a collection of linearly independent cycles C_1, C_2, \dots, C_L such that any other cycle C
 396 can be expressed as a linear combination of cycles in the collection [21].

397 Any connected graph admits a cycle basis. A simple construction follows. First,
 398 pick a spanning tree of the network. Then the spanning tree includes $V - 1$ edges, and
 399 $E - (V - 1)$ edges are left out. The latter are the *chords*. By construction, the tree
 400 does not contain any loops. If one chord is added to the tree then the network contains
 401 exactly one cycle. Note that no two chords can produce the same cycle, and that the
 402 set of cycles produced by adding the chords is necessarily linearly independent since
 403 no chord appears in more than one of these cycles. Let L be the number of chords. If
 404 we enumerate the chords from $1, 2, \dots, L = E - V + 1$ then the set of cycles C_1, \dots, C_L
 405 associated with each chord is a cycle basis. The Figure 4 caption provides an example.

406 A basis generated by a spanning tree is a *fundamental cycle basis* [5, 21]. Cycle
 407 bases are rarely unique, since there are often many possible spanning trees, and not
 408 all bases are fundamental. An alternate basis for the network shown in Figure 4
 409 could be the outer square consisting of edges I, IV, V and III, and either of the triangles.

410 Next, define the cycle space \mathbb{R}^L to be the space of real vectors with one entry for
 411 each cycle in a chosen cycle basis. The dimension of the cycle space $L = E - V + 1$ is
 412 the *cyclomatic number* of the network [5, 21]. We define the *discrete curl* operator
 413 to be the matrix which maps from \mathbb{R}^E to \mathbb{R}^L (edges to cycles) by performing the path
 414 sum around each loop. If $\{k_1, k_2, \dots, k_{|C_l|}\} = C_l$ then:

415 (3.9)
$$[Cf]_l = \sum_{k \in C_l} f_k.$$

416 Note that in order to perform this sum, each loop must be assigned an arbitrary
 417 direction of traversal. This is another sign convention.

space spanned by the vector $[1; 1; \dots; 1]$. It follows that $G(r + c) = Gr$ if c is some constant. This motivates the constraint $\sum_i r_i = 0$ used throughout.

418 We limit our attention to curl operators such that there exists an invertible matrix
419 T for which $TC = \tilde{C}$, where \tilde{C} is the curl defined with respect to a fundamental basis.

420 This curl is analogous to the curl in continuous space, which is a path integral
421 over infinitesimally small loops. Note that the discrete curl defined in this way is
422 more general than the discrete curl defined in [32, 50], where the curl is restricted
423 to act on triangles. Restricting the curl can lead to unintuitive conclusions. For
424 example, if $p_{AB} = p_{BC} = p_{CD} = p_{DA} = 0.99$ then there is clearly a cyclic tendency
425 in competition, but if the curl is restricted to only act on triangles, then the curl
426 would be zero. Here we extend the curl to act on loops of arbitrary length since, like
427 [68], we do not see a fundamental distinction between cyclic structure on triangles
428 and cyclic structure on larger loops. If desired, we could partition the curl operator
429 into blocks, each according to loops of a fixed length, and treat each block as the curl
430 operator restricted to loops of a given size. In this way our approach is distinct from
431 the approaches developed from cohomology, and is closer to the methods developed
432 by Kirchoff to study electric circuits [5].

433 **Figure 4** provides examples of these operators.

434 **LEMMA 3.3 (Orthogonality).** *The curl C and gradient G satisfy $CG = 0$.*

435 *Proof.* Consider the product CGu for some arbitrary vector $u \in \mathbb{R}^V$. The product
436 Gu produces a perfectly transitive edge flow, so the product CGu evaluates the path
437 sum of that edge flow around a set of loops. All perfectly transitive edge flows are
438 arbitrage free, so the path sum of Gu over any loop is zero. It follows that $CGu = 0$
439 for all $u \in \mathbb{R}^V$ so:

$$440 \quad (3.10) \quad CG = 0. \quad \square$$

441 **LEMMA 3.4.** *Let f be an edge flow, C be a curl operator, and G be the gradient.*
442 *If $Cf = 0$, then there exists a set of ratings r such that $Gr = f$.*

443 *Proof.* This Lemma is a direct consequence of **Lemma 3.1**. If C is a curl operator,
444 then there exists an invertible transform T such that $C = T\tilde{C}$ where \tilde{C} is the curl
445 operator with respect to some fundamental cycle basis. Then $Cf = T\tilde{C}f = 0$ if and
446 only if $\tilde{C}f = 0$. Since \tilde{C} is defined with respect to a fundamental cycle basis, \tilde{C} is
447 defined with respect to a spanning tree \mathcal{T} which generates the cycle basis. Requiring
448 that $\tilde{C}f = 0$ is equivalent to requiring that the sum of f around every loop formed by
449 adding one chord into the tree is zero. This condition is sufficient to reconstruct r such
450 that $Gr = f$ using the spanning tree construction given in the proof of **Lemma 3.1**,
451 where the chosen tree is \mathcal{T} . \square

452 **Lemma 3.3** and **Lemma 3.4** establish that, if the edge flow is the gradient of some
453 set of ratings then its curl is zero, and if the curl of the edge flow is zero then it can be
454 expressed as the gradient of some set of ratings. Therefore the range of the gradient is
455 the nullspace of the curl. The equivalence of these two spaces and the orthogonality of
456 the operators allows us to decompose f into unique perfectly transitive and perfectly
457 cyclic components. This decomposition is the HHD.

458 **3.2.2. The Discrete Helmholtz-Hodge Decomposition.**

459 **THEOREM 3.5 (The HHD).** *Any $f \in \mathbb{R}^E$ can be decomposed such that:*

$$460 \quad (3.11) \quad f = f_t + f_c$$

461 *where f_t is arbitrage free (perfectly transitive) and f_c is favorite free (perfectly cyclic)*
462 *and both are unique. In addition, there exists a unique rating r satisfying $\sum_i r_i = 0$*

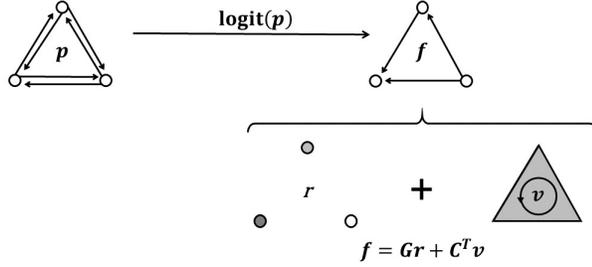


FIG. 5. A schematic representation of the decomposition for a complete tournament on three competitors. The edge flow f is set equal to $\text{logit}(p)$, and then broken into a set of ratings r and vorticities v , such that $f = Gr + C^T v$.

463 such that $f_t = Gr$ and, for any choice of C , a unique vorticity $v \in \mathbb{R}^L$ exists such
 464 that $f_c = C^T v$. Thus the original edge flow f can be uniquely decomposed:

$$465 \quad (3.12) \quad f = Gr + C^T v.$$

466 *Proof.* By the fundamental theorem of linear algebra $\mathbb{R}^E = \text{null}(C) \oplus \text{range}(C^T)$
 467 [74]. Lemma 3.3 and Lemma 3.4 guarantee that $\text{range}(G) = \text{null}(C)$, so:

$$468 \quad (3.13) \quad \mathbb{R}^E = \text{range}(G) \oplus \text{range}(C^T).$$

469 Thus any edge flow can be uniquely decomposed into the sum of a perfectly transitive
 470 and perfectly cyclic edge flow, and those edge flows are the projections of f onto the
 471 perfectly transitive and cyclic subspaces.

472 Equation (3.13) establishes that there exists an r such that $Gr = f_t$, and a v such
 473 that $C^T v = f_c$. We have already proved r was unique. Equation (3.13) guarantees
 474 $E = \text{rank}(G) + \text{rank}(C^T)$. In general, G has nullity equal to the number of connected
 475 components in the network. We assumed the network is connected, so G has a one-
 476 dimensional nullspace and rank $V - 1$. Therefore, $\text{rank}(C^T) = E - (V - 1) = L$.
 477 By construction, C^T has L columns, so is full rank. It follows that the linear system
 478 $C^T v = f$ has a unique solution if $f \in \text{range}(C^T)$. \square

479 Therefore, any arbitrary tournament can be decomposed into a perfectly transitive
 480 and a perfectly cyclic tournament, where the perfectly transitive tournament is
 481 specified by a set of ratings, and the perfectly cyclic tournament is specified by a set
 482 of vorticities. The ratings associated with the HHD are the Hodge ratings proposed
 483 by [32]. Figure 5 provides a schematic representing the decomposition.

484 The three example networks displayed in Figure 2 are actually an example of
 485 an HHD. Reading left to right, the first network is perfectly transitive, the second
 486 is perfectly cyclic, and they add to produce the generic network shown on the right.
 487 The edge flows, ratings r , and vorticities v are shown in Figure 6.

488 The gradient G has exactly 2 nonzero entries per edge, so it becomes sparser as the
 489 number of competitors increases. Consequently, the decomposition can be performed
 490 efficiently, even for large, fully connected networks. Methods are discussed in [10, 32].

491 The intransitivity measure associated with the HHD is the size of the cyclic component
 492 $\|f_c\|_2$. Because the HHD is a decomposition onto orthogonal subspaces, this
 493 measure is equal to the distance from f to the closest perfectly transitive tournament.
 494 Therefore the Helmholtz-Hodge intransitivity measure is conceptually analogous to
 495 the Slater intransitivity measure [68], and its variants [57], [70], [77]. Similarly, the

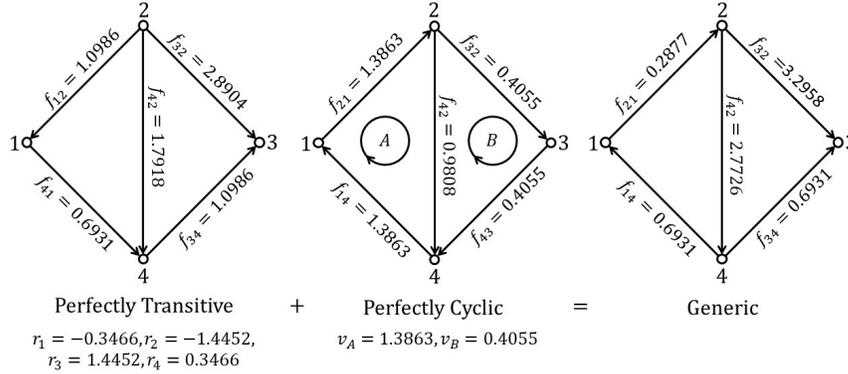


FIG. 6. An example HHD using the three networks from Figure 2. From left to right: the leftmost network is perfectly transitive, the middle network is perfectly cyclic, and the network on the right is the sum of the perfectly transitive and cyclic networks. The ratings associated with the perfectly transitive graph are provided beneath it. Notice that the difference in the ratings recover the edge flow on each edge. For example, $r_3 - r_4 = 1.4452 - 0.3466 = 1.0986 = f_{34}$. Also notice that the curl of the edge flow around any loop is zero. For example, $f_{41} + f_{12} = 0.6931 + 1.0986 = 1.7918 = f_{42}$ so $f_{41} + f_{12} + f_{24} = f_{41} + f_{12} - f_{42} = 0$. The vorticities associated with the perfectly cyclic network are provided beneath it. Notice that the perfectly cyclic edge flow satisfies the neighborhood condition. For example, the total flow into node 2 is $1.3863 - 0.9808 - 0.4055 = 0$. Finally, notice that the values of the edge flow in the rightmost network are the sum of the edge flows in the perfectly transitive and cyclic networks. For example, looking at the edge connecting nodes 1 and 2, $-1.0986 + 1.3863 = 0.2877$.

496 transitivity measure associated with the HHD is the size of the transitive component
 497 $\|f_t\|_2$, and is the distance from f to the closest perfectly cyclic tournament.

498 Note that these measures are continuous in p . In contrast, classical methods such
 499 as the Kendall [35] or Slater [68] measures only depend on \mathcal{G}_\rightarrow so are discrete in
 500 p . This distinction is important, since it means that the Helmholtz-Hodge measure
 501 distinguishes between the cases $p_{AB} = p_{BC} = p_{CA} = 0.99$ and $p_{AB} = p_{BC} = p_{CA} =$
 502 0.51 (intransitivity 7.96 and 0.07 respectively). Using the discrete measures, these two
 503 tournaments are equally intransitive. Thus the Helmholtz-Hodge measure reflects the
 504 absolute strength of cyclic competition by distinguishing strong and weak cycles. The
 505 discrete measures reflect the relative strength of cyclic competition since they only
 506 depend on the sign of f , which depends on both f_c and f_t . If the transitive part is
 507 large then it may mask weaker cyclic competition when using a discrete measure. For
 508 example, if $p_{AB} = 0.99$, $p_{BC} = 0.99$ and $p_{CA} = 0.49$ then the probability that C beats
 509 A is much larger than might be expected. However, in this example competition is
 510 transitive so all discrete measures of intransitivity would return their minimal value, 0.
 511 In contrast, the Helmholtz-Hodge measure returns intransitivity 5.29. These examples
 512 are illustrated in Figure 7. Normalizing the Helmholtz-Hodge measures by $\|f\|_2$
 513 produces the equivalent relative measures: $\|f_c\|_2/\|f\|_2$ and $\|f_t\|_2/\|f\|_2$.

514 **3.2.3. Equivalent Formulations.** Here we present six different approaches
 515 that arrive at the same decomposition. These provide different, useful, perspectives on
 516 the HHD, and illustrate that it is robust to varying motivations. The ensuing Corol-
 517 lary follows directly from standard properties of projection onto orthogonal subspaces,
 518 so we omit the proof.

519 **COROLLARY 3.6 (Equivalent Formulations).** *The following six decompositions*
 520 *are equivalent:*

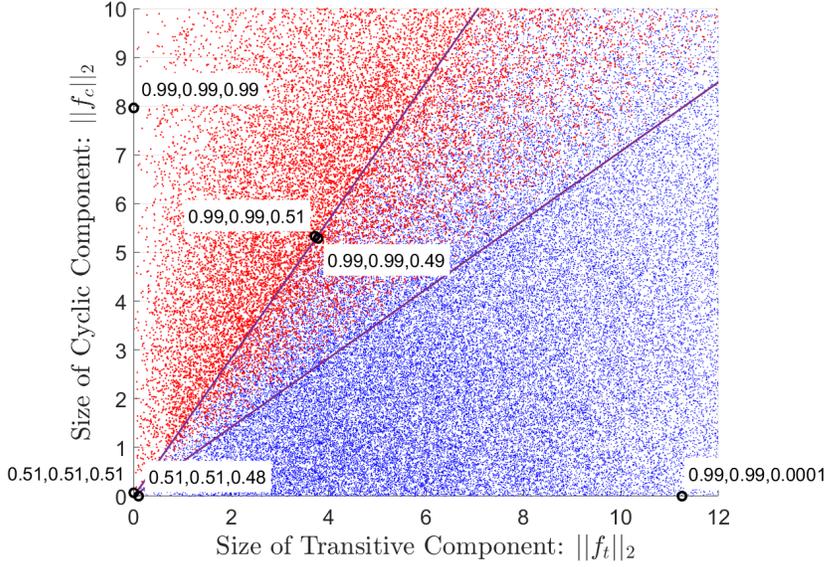


FIG. 7. Transitivity and intransitivity of 10^4 triangular networks with randomly drawn win probabilities. The horizontal axis is the size of the transitive component and the vertical axis is the size of the cyclic component. Each scatter point is a sampled network. Smaller blue scatter points are transitive, larger red points are intransitive. The upper and lower purple lines (slope $\sqrt{2}$ and $\sqrt{0.5}$) divide regions where competition on triangles is always cyclic, either transitive or cyclic, and always transitive. The large black circles represent example networks. The text next to each example gives the probability A beats B, B beats C, and C beats A. If all of these numbers are greater than 0.5 then the network is intransitive. Note that the classification into transitive and intransitive draws a sharp distinction between networks whose win probabilities are nearly identical, while networks with similar win probabilities remain close to each other when using the Hodge measures. Also note that the boundary between transitive and intransitive networks is an angular sector, hence this classification is based on the relative sizes of the transitive and cyclic components, not their absolute sizes. In contrast the Hodge measures reflect the absolute size of each component. Thus the example with win probabilities 0.99, 0.99, 0.49 can be transitive and the example 0.51, 0.51, 0.51 can be intransitive, even though the former has a larger cyclic component than the latter.

- 521 1. $f = f_t + f_c$ where f_t is arbitrage free and f_c is favorite free;
- 522 2. $f = f_t + f_c$ where $f_t = Gr$ for ratings r and $f_c = C^T v$ for vorticity v ;
- 523 3. the ratings r solve the constrained least squares problem:

524 (3.14) Minimize $\|Gu - f\|_2^2$ given $u \in \mathbb{R}^V$ and $\sum_{i=1}^V u_i = 0$

- 525 and $f_t = Gr, f_c = f - f_t$;
- 526 4. the vorticities v solve the least squares problem:

527 (3.15) Minimize: $\|C^T w - f\|_2^2$ given $w \in \mathbb{R}^L$

- 528 and $f_c = C^T v, f_t = f - f_c$;
- 529 5. $f = f_t + f_c$ where $f_t = Gr$ for the unique ratings r such that the circulant
- 530 $f - f_t$ is favorite free;
- 531 6. $f = f_t + f_c$ where $f_c = C^T v$ for the unique vorticities v such that $f - f_c$ is
- 532 arbitrage free.

533 Each of these approaches provides a different perspective on the HHD. We might
 534 seek to decompose f into components that do not circulate and do not converge, into
 535 components defined by a set of ratings and vorticities, according to the best perfectly
 536 transitive or perfectly cyclic approximation, so that the residue left over when ap-
 537 proximating f does not circulate, or so that the residue left over when approximating
 538 f does not converge anywhere. In each case the resulting decomposition is the same.
 539 The fact that the HHD is equivalent to all of these approaches motivates its use.

540 It is worth highlighting the third and fourth approach, which show that f_t is
 541 the nearest perfectly transitive edge flow to f , and f_c is the nearest perfectly cyclic
 542 edge flow to f . Decomposition 3 shows that the ratings produced by the HHD are
 543 a type of least squares rating. Least squares ratings methods are widely used [6, 14,
 544 34, 41, 42, 45, 51, 72, 73]. Although the literature has focused almost exclusively on
 545 Decomposition 3, Decompositions 3 and 4 are dual to one another. This parity in
 546 approach sets the HHD apart from existing methods.

547 **4. Null Models and the Trait-Performance Theorem.** How intransitive is
 548 a typical tournament?

549 Answering this question requires defining a statistical model for sampling tour-
 550 naments - in particular, for sampling edge flows. How do assumptions about the
 551 distribution of possible edge flows affect the expected strength of cyclic competition?
 552 What statistical features tend to promote or suppress cyclic competition?

553 We initially explore these questions for a generic null model where the edge flow,
 554 F , is sampled randomly from an unspecified distribution. This analysis identifies
 555 which features of the edge flow and the network topology influence the degree of
 556 cyclic competition. These conclusions set the stage for the following insight.

557 If the edge flow is sampled using a trait-performance model, then the covariance
 558 of the edge flow takes on a canonical form which depends only on *two* statistical quan-
 559 tities: the variance in the flow on each edge, and the correlation in the flow on pairs
 560 of edges that share an endpoint. This simplified structure leads to an elegant closed
 561 form expression for the expected sizes of the cyclic and transitive components that
 562 separates the influence of the network topology from the trait-performance statistics.

563 We generalize this result in two ways. First, the relations between correlation
 564 and network structure derived under the trait-performance assumptions hold for any
 565 complete network - whether or not the trait-performance assumptions are valid. Sec-
 566 ond, we show that the canonical form for the covariance can be used to design null
 567 models for tournaments with tunable transitive structure. These models can be easily
 568 adjusted to promote or suppress cycles, and could be used to define more nuanced
 569 transitivity tests than the standard randomization tests [2, 15, 35].

570 **4.1. Generic Null Models.** We start by considering generic null models where
 571 the edge flow $F \in \mathbb{R}^E$ is drawn randomly from some distribution. For now we in-
 572 troduce no assumptions on the distribution other than that it has finite first and
 573 second moments. Denote the expected edge flow $\bar{f} = \mathbb{E}[F]$ and the covariance
 574 $\text{Cov}(F) = \mathbb{E}[(F - \bar{f})(F - \bar{f})^T]$.

575 Let P_c be the orthogonal projector onto the space of perfectly cyclic (favorite
 576 free) tournaments. Then the expected squared strength of cyclic competition is:

$$\begin{aligned}
 \mathbb{E}[||F_c||^2] &= \mathbb{E}[F^T P_c^T P_c F] = \mathbb{E}[F^T P_c F] = \sum_{kl} (P_c)_{kl} \mathbb{E}[F_k F_l] \\
 (4.1) \quad &= \sum_{kl} (P_c)_{kl} (\bar{f}_k \bar{f}_l + \text{Cov}(F)_{kl}) = ||\bar{f}_c||^2 + \text{trace}(P_c \text{Cov}(F))
 \end{aligned}$$

578 where $\|\bar{f}_c\|^2 = \bar{f}^T P_c \bar{f}$ is the cyclic component of the expected edge flow.

579 Therefore, no matter the underlying distribution of edge flows, the expected
580 strength of cyclic competition is determined exclusively by three quantities: the *ex-*
581 *pected edge flow*, the *covariance in the edge flow*, and the *topology of the network*
582 (which determines P_c).

583 The matrix inner product, $\text{trace}(P_c \text{Cov}(F))$, can be simplified if the flows on each
584 edge are independent. Then $\text{Cov}(F)$ is diagonal with entries $\sigma_k^2 = \mathbb{E}[(F_k - \bar{f}_k)^2]$. It
585 follows that $\text{trace}(P_c V) = \sum_{k=1}^E (P_c)_{kk} \sigma_k^2$.

586 The nonzero eigenvalues of a projector all equal one, so its trace equals the dimen-
587 sion of the space it projects onto. The projector P_c projects onto the space of perfectly
588 cyclic tournaments, which has dimension $L = E - (V - 1)$. Therefore $\sum_k (P_c)_{kk} = L$.
589 Rewrite the expected strength of cyclic competition:

$$590 \quad (4.2) \quad \mathbb{E}[\|F_c\|^2] = \|\bar{f}_c\|^2 + L \sum_{k=1}^E \left(\frac{(P_c)_{kk}}{L} \right) \sigma_k^2.$$

591 Since the diagonal entries of an orthogonal projector are always nonnegative, the
592 right hand term can be interpreted as a weighted average of the variance on each
593 edge. Therefore, when the edges are independent, the expected strength of cyclic
594 competition is given by the strength of the cyclic component of the expected edge
595 flow, plus the dimension of the loop space times a weighted average of the variance
596 on each edge. Similarly, the expected strength of transitive competition is:

$$597 \quad (4.3) \quad \mathbb{E}[\|F_t\|^2] = \|\bar{f}_t\|^2 + (V - 1) \sum_{k=1}^E \left(\frac{(P_t)_{kk}}{V - 1} \right) \sigma_k^2$$

598 and the expected total strength of competition is:

$$599 \quad (4.4) \quad \mathbb{E}[\|F\|^2] = \|\bar{f}\|^2 + E\bar{\sigma}^2$$

600 where $\bar{\sigma}^2$ is the average of the variance in the flow on each edge. Equation (4.4) is
601 valid even if the edges are not independent, as the projector onto the full space is
602 simply the identity.

603 Equations (4.2) - (4.4) show that the contribution to the expected strength of
604 competition from the variances is not distributed equally between the transitive and
605 cyclic spaces. Instead, the amount that is cyclic is proportional to the number of
606 cycles, while the amount that is transitive is proportional to the number of com-
607 petitors. As a result, adding edges to a network will typically increase the expected
608 degree to which competition is cyclic. It follows that sparse networks with randomly
609 drawn edge flows will be relatively more transitive than would be expected given \bar{f} ,
610 while dense networks will typically be more cyclic. It also follows that, for a posterior
611 distribution of possible edge flows given observed data, uncertainty will likely lead
612 to an overestimate of the degree to which competition is cyclic when the network is
613 dense. If a tournament is complete, then $E = V(V - 1)/2$ so $(V - 1)/E = 2/V$ and
614 $L/E = 1 - 2/V$. It follows that, for a complete tournament with more than four com-
615 petitors, any uncertainty in the edge flow will typically bias competition to appear
616 more cyclic than transitive.⁹

⁹This result does not contradict Shizuka's result that the proportion of transitive triangles in a network with uniformly randomly sampled dominance relations is independent of the network topology [63], since our measure accounts for the global structure of the edge flow, thus incorporates cyclic structure over longer cycles.

617 Numerical studies have suggested that filling in missing edges with randomly
 618 drawn F typically overestimates the degree to which competition is cyclic, thereby
 619 weakening transitivity tests [63]. Our result provides a clear explanation for this
 620 observation. When the edge flow F is drawn randomly to fill in missing data, it is
 621 usually drawn independently and identically distributed, cf. [15]. If edges are added
 622 until the network is complete, then, for any tournament with more than four com-
 623 petitors, the resulting “imputed” tournament will likely be significantly more cyclic
 624 than the original tournament. Therefore, unless the edge flows are well modeled by
 625 assuming that the F_k are independent and identically distributed, *and* that all pairs of
 626 competitors could compete, this procedure is not valid for estimating the strength of
 627 cyclic competition in a partially observed tournament. This observation underscores
 628 the need for intransitivity measures that can be applied to incomplete tournaments.

629 Unfortunately the projectors P_t and P_c may be expensive to compute, and can-
 630 not always be constructed directly without performing a matrix decomposition. This
 631 makes it challenging to identify exactly how the topology of the network and covari-
 632 ance structure promote or suppress cyclic competition. Nevertheless, as we show in
 633 the next section, using a more principled model for sampling F , ensures that the co-
 634 variance matrix $\text{Cov}(F)$ takes on a canonical form. This form clarifies the interaction
 635 between the topology of the network and the distribution of edge flows.

636 **4.2. Trait-Performance.** The outcomes of real-world competition events are
 637 typically influenced by a constellation of underlying competitor traits. Examples of
 638 trait-influenced competition abound, ranging from sports¹⁰ to simulated competitive
 639 events to biology.¹¹ In some cases, trade-offs inherent in certain traits have been ob-
 640 served to lead to cyclic competition between organisms [36, 66].¹² In such examples,
 641 trade-offs lead to advantages against certain opponents, and weaknesses that are ex-
 642 ploited by others. In evolutionary biology, trade-offs of this kind challenge the notion
 643 that members of intransitive communities can be consistently ranked according to fit-
 644 ness. Intransitivity can lead to deeply counterintuitive evolutionary dynamics [20, 33],
 645 and may promote biodiversity since no single species has an absolute advantage over
 646 all competitors [59, 58, 60, 61, 70]. These considerations motivate a study of how the

¹⁰Some predictive tennis models estimate the probability that one competitor will beat another based on a parameterized model for the probability that each player will win a point, where the underlying parameters depend on traits of the players [40]. Similarly, considerable effort has been devoted to predictive models for baseball based on team and player statistics [78].

¹¹Ecological studies of competition for dominance in social hierarchies have analyzed how traits confer success, because selection acts on heritable traits contributing to reproductive success. Examples include competition among male northern elephant seals [26] and male Cape dwarf chameleons [76]. Relevant traits for elephant seals include body mass, length, age, and time of arrival on the beach [26]. Relevant traits for chameleons include body mass, length from snout to base of tail, length of the tail, jaw length, head width, casque size, and size of a pink colored flank patch used in signaling [76].

¹²Two particularly famous examples are side-blotched lizards and colicin producing *E. coli* [36, 66]. In the former example, large orange-throated males maintain large territories, medium blue-throated males defend small territories, while small yellow-throated ‘sneaker’ males resemble females and do not maintain territories. Orange-throated males typically defeat the smaller blue-throated males, who defeat the even smaller yellow throated males, who defeat the orange throated males by sneaking into their territories [66]. In the latter example, three strains of *E. coli* were grown in direct competition in a laboratory setting. The first strain produced a colicin toxin, the second was susceptible to the toxin, and the third was resistant to the toxin but not toxin-producing. In the absence of the resistant strain, the toxic strain could outcompete the susceptible strain. In the absence of the toxic strain, the susceptible strain could outcompete the resistant strain, which reproduced more slowly because resistance is costly. But, in the absence of the susceptible strain, the resistant strain could outcompete the toxic strain by reproducing more quickly [36].

647 distribution of traits, and the way traits confer success, either promote or suppress
648 cyclic competition.

649 To study this scenario, suppose that win probabilities p can be modeled as a
650 function of some underlying traits x of each competitor. Let $X(i) = [X_1(i), \dots, X_T(i)]$
651 denote the T randomly sampled traits of the i^{th} competitor. Then let $f(x, y)$ be a
652 performance function, such that $f(x, y)$ is the log-odds that a competitor with traits
653 x would beat a competitor with traits y .

654 To construct a trait-performance model assume that:

- 655 1. The trait vectors of the competitors are drawn independently and identically
656 from a trait distribution π_x .
- 657 2. There exists a performance function $f(x, y)$ that maps from $\mathbb{R}^T \times \mathbb{R}^T$ to \mathbb{R} .
658 We require that the performance function is alternating, $f(x, y) = -f(y, x)$,
659 for any trait vectors x and y in the support of π_x . This ensures that f
660 can be used to generate an edge flow. It also ensures that the performance
661 function is fair, $\mathbb{E}[f(X, Y)] = 0$, since when X and Y are drawn i.i.d then
662 $\mathbb{E}[f(X, Y)] = \mathbb{E}[f(Y, X)] = -\mathbb{E}[f(X, Y)]$ which implies $\mathbb{E}[f(X, Y)] = 0$.
- 663 3. There exists a connected competitive network $\mathcal{G}_{\rightleftharpoons}$ with edges representing
664 possible competition events, and the network is either fixed a priori or sampled
665 independently from the traits.

666 The first assumption holds if all competitors are drawn from the same trait pool.
667 Different pools can be incorporated into the model by adding a trait which indexes
668 which pool each competitor is sampled from, provided that trait can be sampled
669 independently of the graph. For example, Major League Baseball team budgets vary
670 widely. In 2018 the Yankees' total value was over 4.6 billion dollars, which was
671 more than the total value of the bottom six teams combined [56]. This difference in
672 resources gives high value teams the opportunity to pay higher salaries¹³ and attract
673 stars. Thus wealth could be incorporated as a trait.

674 The second assumption is valid whenever the probability that one competitor
675 beats another can be conditioned on the traits of the competitors, independent of
676 their location on the network, and of the outcomes of past events. Note that in some
677 biological contexts, such as social hierarchies, event outcomes are not necessarily
678 independent, and may be influenced by past events. For example, winner, loser, and
679 bystander effects, in which winners are more likely to win again, losers are more likely
680 to lose again, and bystander behavior is influenced by observed events between other
681 competitors, play an important role in the self-organization of certain social hierarchies
682 [12, 13, 28, 55, 65]. The assumption that competition outcomes are mediated by traits
683 is also not supported in convention based societies where rank is determined by a
684 social convention, such as matrilineal rank inheritance (c.f. [69, 75]). Nevertheless,
685 other hierarchies can be explained by traits (c.f. [31, 62]), and even in situations when
686 competition outcomes are influenced by past events, competitor attributes typically
687 influence competition outcomes as well [4, 13].

688 The third assumption treats the network topology (who competes with whom) as
689 independent from the traits of the competitors. This may not be realistic if competi-
690 tors avoid competing when they are likely to lose [67]. This also limits our ability to
691 model systems where traits or rank are heritable (c.f. [69, 75]), or distributed differ-
692 ently across different clusters of competitors (different divisions, or local populations).

693 While these assumptions do not hold in all situations, they provide a tractable
694 paradigm that lays the foundation for a more general understanding.

¹³For example, in 2019 the Yankees' combined payroll was three times larger than the Marlins'.

695 Under assumptions 1-3, we define a trait-performance model as follows. First,
 696 sample $X(i) \sim \pi_x$ for all competitors i . Then, set $F_k = f(X(i(k)), X(j(k)))$, where
 697 $i(k), j(k)$ are the endpoints of edge k .

698 **THEOREM 4.1 (Trait-Performance).** *Let $\mathcal{G}_{\rightleftharpoons}$ be a competitive network with V*
 699 *competitors, E edges and L loops, satisfying assumption 3. If the traits of each*
 700 *competitor are drawn independently from π_x , and the edge flow is defined by $F_k =$*
 701 *$f(X(i(k)), X(j(k)))$ where $f(x, y)$ is an alternating performance function, then the*
 702 *covariance $\text{Cov}(F)$ of the edge flow has the form:*

$$703 \quad (4.5) \quad \text{Cov}(F) = \sigma^2 [I + \rho (GG^T - 2I)]$$

704 where σ^2 is the variance in F_k for arbitrary k , and ρ is the correlation coefficient
 705 between $f(X, Y)$ and $f(X, W)$ for X, Y, W drawn i.i.d from π_x .

706 *Moreover:*

$$707 \quad (4.6) \quad \mathbb{E} \left[\frac{1}{E} \|F\|^2 \right] = \sigma^2 \xrightarrow{\text{decompose}} \begin{cases} \mathbb{E} \left[\frac{1}{E} \|F_t\|^2 \right] = \sigma^2 \left[\frac{(V-1)}{E} + 2\rho \frac{L}{E} \right] \\ \mathbb{E} \left[\frac{1}{E} \|F_c\|^2 \right] = \sigma^2 (1 - 2\rho) \frac{L}{E} \end{cases}$$

708 *The correlation ρ ranges from 0 to 1/2, and if $\rho = 1/2$ then competition is perfectly*
 709 *transitive.*

710 *Proof.* First consider the covariance matrix $\text{Cov}(F)$.

711 Since the trait vectors are drawn i.i.d from the trait distribution, the diagonal
 712 entries of the covariance are given by:

$$713 \quad (4.7) \quad \text{Cov}(F)_{kk} = \mathbb{E} [f(X(i(k)), X(j(k)))^2] = \mathbb{E} [(f(X, Y))^2] \equiv \sigma^2$$

714 where X, Y are drawn i.i.d from the trait distribution, and σ^2 is the variance in
 715 $f(X, Y)$. Thus, the diagonal entries of the covariance are identical.

716 The off-diagonal entries are $\mathbb{E} [f(X(i(k)), X(j(k))) \cdot f(X(i(l)), X(j(l)))]$.

717 Suppose the edges k and l do not share an endpoint. Then $i(k) \neq i(l)$ or $j(l)$
 718 and $j(k) \neq i(l)$ or $j(l)$. Then $f(X(i(k)), X(j(k)))$ is a function of two random vec-
 719 tors, and $f(X(i(l)), X(j(l)))$ is a function of two other random vectors, where the
 720 pair of random vectors are independent. It follows that $f(X(i(k)), X(j(k)))$ is inde-
 721 pendent of $f(X(i(l)), X(j(l)))$. Then, since competition is fair for all alternating
 722 performance functions, $\text{Cov}(F)_{kl} = \mathbb{E} [f(X(i(k)), X(j(k))) \cdot f(X(i(l)), X(j(l)))] =$
 723 $\mathbb{E} [f(X(i(k)), X(j(k)))] \mathbb{E} [f(X(i(l)), X(j(l)))] = 0$. It follows that the support of the
 724 covariance matches the adjacency structure of the edges of the competition network.

725 If the edges do share an endpoint, then there are four possibilities. Either $i(k) =$
 726 $i(l)$, $j(k) = j(l)$, $i(k) = j(l)$, or $j(k) = i(l)$. We say that the edges are *consistently*
 727 *oriented* if they share either the same starting point or the same ending point, and
 728 are *inconsistently oriented* if the endpoint of one is the start of another. Since all the
 729 trait vectors are drawn i.i.d., we suppress the indices and let the three trait vectors
 730 Y, W, Z be drawn i.i.d. from π_x . The performance function is alternating, so:

$$731 \quad (4.8) \quad \begin{aligned} \mathbb{E}[f(Y, W)f(Y, Z)] &= \mathbb{E}[f(W, Y)f(Z, Y)] \equiv \rho\sigma^2 \\ \mathbb{E}[f(Y, W)f(Z, Y)] &= \mathbb{E}[f(W, Y)f(Y, Z)] = -\mathbb{E}[f(Y, W)f(Y, Z)] = -\rho\sigma^2 \end{aligned}$$

732 where ρ is the correlation coefficient between $f(Y, W)$ and $f(Y, Z)$. Notice that a
 733 positive correlation indicates that the probability that A beats B is increased by
 734 conditioning on the event that A beats C .

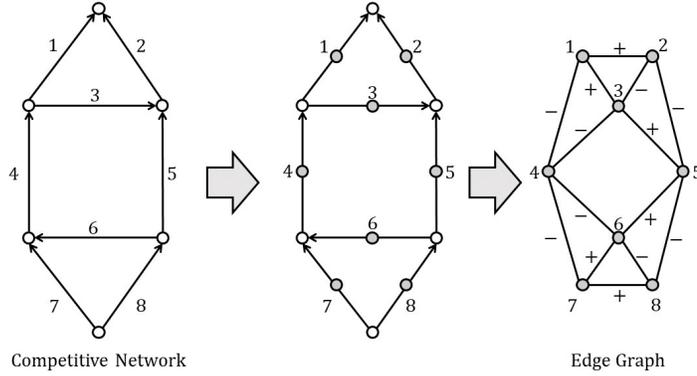


FIG. 8. The edge graph (right) associated with a competitive network (left). The middle panel shows an intermediate graph where a node has been introduced for each edge. The edges of the competitive network become the nodes of the edge graph. The edges of the edge graph correspond to nodes in the competitive network that are the shared endpoint of a pair of edges. These are labelled with a + or - to indicate whether the edges are consistently or inconsistently oriented with respect to the shared endpoint.

735 The *edge graph* is the graph with a node for each edge in the competition network,
 736 and with an undirected edge between nodes corresponding to connected edges in the
 737 competition network (Figure 8). Let A_E be the weighted adjacency matrix for the
 738 edge graph with $a_{Ekl} = +1$ or -1 if edges k and l are consistently or inconsistently
 739 oriented with respect to a shared endpoint. Then:

740 (4.9)
$$\text{Cov}(F) = \sigma^2 [I + \rho A_E].$$

741 The weighted adjacency matrix A_E for the edge graph is equal to $GG^T - 2I$ since:

742 (4.10)
$$[GG^T]_{kl} = (e_{i(k)} - e_{j(k)})^T (e_{i(l)} - e_{j(l)}) = \begin{cases} 2 & \text{if } k = l \\ 1 & \text{if } i(k) = i(l) \text{ or } j(k) = j(l) \\ -1 & \text{if } i(k) = j(l) \text{ or } j(k) = i(l) \\ 0 & \text{else} \end{cases}$$

743 where $e_i \in \mathbb{R}^V$ is the indicator vector for node i . Thus we establish (4.5).

744 All of the absolute measures of the strength of competition (squared) are given by
 745 the squared length of the orthogonal projection of the edge flow onto some subspace.
 746 Let P_S be an arbitrary orthogonal projector onto some subspace S . By construction,
 747 the edge flow is zero mean, therefore, by equation (4.1), the expected value of the
 748 associated measure is:

749 (4.11)
$$\mathbb{E} [||F_S||^2] = \text{trace}(P_S \text{Cov}(F)).$$

750 The intensity of competition, $||F||^2$, corresponds to the projector I , $||F_t||^2$ cor-
 751 responds to the projector P_t , and $||F_c||^2$ corresponds to the projector P_c . Then, by
 752 equation (4.11):

753 (4.12)
$$\mathbb{E} \left[\frac{1}{E} ||F||^2 \right] = \frac{1}{E} \text{trace}(\text{Cov}(F)) = \frac{E}{E} \sigma^2 = \sigma^2.$$

754 This formula establishes that the absolute strength of competition only depends
 755 on the variance σ^2 in each individual performance function.

756 To compute $\|F_t\|^2$, use equation (4.11) with projector P_t :

$$757 \quad (4.13) \quad \begin{aligned} \mathbb{E} \left[\frac{1}{E} \|F_t\|^2 \right] &= \frac{1}{E} \text{trace}(P_t \text{Cov}(F)) = \frac{\sigma^2}{E} \text{trace}(P_t [I + \rho(GG^T - 2I)]) \\ &= \frac{\sigma^2}{E} \text{trace}(P_t) + \frac{\rho\sigma^2}{E} \text{trace}(P_t(GG^T)) - \frac{2\rho\sigma^2}{E} \text{trace}(P_t). \end{aligned}$$

758 The trace of an orthogonal projector equals the dimension of the subspace it
759 projects onto, so $\text{trace}(P_t) = V - 1$. The range of GG^T is in the range of G , which
760 is the subspace P_t projects onto. It follows that $P_t GG^T = GG^T$ so $\text{trace}(P_t GG^T) =$
761 $\text{trace}(GG^T) = 2E$ (see equation (4.10)). Therefore:

$$762 \quad (4.14) \quad \mathbb{E} \left[\frac{1}{E} \|F_t\|^2 \right] = \sigma^2 \left[\frac{V-1}{E} + 2\rho \frac{E-(V-1)}{E} \right] = \sigma^2 \left[\frac{V-1}{E} + 2\rho \frac{L}{E} \right].$$

763 Since $L \geq 0$, $\mathbb{E}[\frac{1}{E}\|F_t\|^2]$ increases monotonically in ρ : the larger ρ , the more A
764 beating B is correlated with A beating C , implying transitive competition.

765 Then, by the orthogonality of the decomposition $f = f_c + f_t$:

$$766 \quad (4.15) \quad \mathbb{E} \left[\frac{1}{E} \|F_c\|^2 \right] = \mathbb{E} \left[\frac{1}{E} \|F\|^2 \right] - \mathbb{E} \left[\frac{1}{E} \|F_t\|^2 \right] = \sigma^2 [1 - 2\rho] \frac{L}{E}.$$

767 It follows that the expected absolute strength of cyclic competition is monoton-
768 ically decreasing in the correlation coefficient ρ . Note that, as when considering the
769 generic null models, dense networks promote cyclic competition.

770 To conclude, we show that $\rho \in [0, 1/2]$, so the expected measures are maximized
771 and minimized when ρ is 0 or $1/2$, respectively.

772 The correlation ρ is nonnegative since W and Z are i.i.d., thus $f(y, W)$ and $f(y, Z)$
773 are also i.i.d. for all y . Then:

$$774 \quad (4.16) \quad \begin{aligned} \sigma^2 \rho &= \mathbb{E}_{Y,W,Z} [f(Y, W)f(Y, Z)] = \int_{\mathbb{R}^T} \mathbb{E}_{W,Z} [f(y, W)f(y, Z)] \pi_x(y) dy \\ &= \int_{\mathbb{R}^T} \mathbb{E}_W [f(y, W)] \mathbb{E}_Z [f(y, Z)] \pi_x(y) dy = \int_{\mathbb{R}^T} \mathbb{E}_W [f(y, W)]^2 \pi_x(y) dy \geq 0. \end{aligned}$$

775 Here expectation is taken with respect to the variables in the subscript.

776 To prove that $\rho \leq 1/2$, note that all covariance matrices are positive semi-definite,
777 so, for any vector u :

$$778 \quad (4.17) \quad u^T \text{Cov}(F)u = \sigma^2 u^T (I + \rho(GG^T - 2I))u = \sigma^2 (1 - 2\rho) \|u\|^2 + \rho u^T GG^T u \geq 0.$$

779 If $E > V - 1$, then the network has at least one loop, so the range of C^T is
780 non-empty, hence the null-space of G^T is non-empty. Choosing u perfectly cyclic sets
781 $G^T u = 0$ so $\sigma^2 (1 - 2\rho) \|u\|^2 \geq 0$ which requires $\rho \leq \frac{1}{2}$. If $E = V - 1$ then the network
782 is a tree, so all competition is necessarily perfectly transitive.

783 It follows that the expected absolute strength of *transitive* competition is mini-
784 mized when $\rho = 0$, and maximized when $\rho = 1/2$. In contrast, the expected strength
785 of *cyclic* competition is maximized when $\rho = 0$, and minimized when $\rho = 1/2$.

786 If $\rho = 1/2$ then $\mathbb{E}[\|F_c\|^2] = 0$. The measure is nonnegative for all edge flows.
787 Therefore, its expected value is only zero if the probability that $\|F_c\|^2 \neq 0$ is zero.

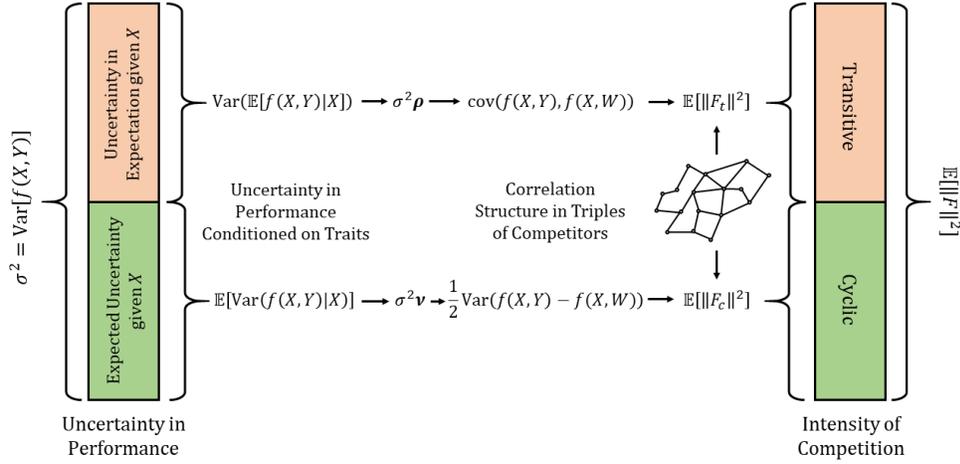


FIG. 9. A schematic representing the conclusions of *Theorem 4.1* and *Corollary 4.2*. The left hand side decomposes the uncertainty in performance into the uncertainty in the expected performance given X , and the expected uncertainty in the performance, given X . These uncertainties are converted into ρ and ν which describe the correlation structure of triples of competitors. The sizes of ρ and ν , plus the topology of the network, determine the expected sizes of the transitive and cyclic components. Thus we convert a decomposition of the uncertainty in the performance into a decomposition of the intensity of the edge flow representing competition.

788 In this case, the tournament is arbitrage free. It follows that, if $\rho = 1/2$, then the
 789 tournament must be perfectly transitive.¹⁴ \square

790 *Theorem 4.1* establishes that the expected degree to which competition is transitive
 791 or cyclic depends principally on the density of the network, and the correlation
 792 structure of F . In particular, the degree to which a network is cyclic or transitive
 793 depends on the correlation between the performance of A against B with the per-
 794 formance of A against C . The larger this correlation, the more consistently each
 795 competitor performs, hence the more consistent the network is with a set of ratings.

796 The variance σ^2 and the correlation coefficient ρ could be computed given an
 797 assumed trait distribution π_x and performance function $f(x, y)$. This could be done
 798 analytically if π_x and f lead to simple calculations. Otherwise, σ^2 and ρ can be
 799 approximated numerically by sampling or quadrature. The analytic method follows.

800 Suppose that X, Y are drawn from a sample space Ω which is a subset of \mathbb{R}^T .
 801 Then, for trait distribution π_x :

$$802 \quad (4.18) \quad \rho = \frac{\mathbb{E}_X[\mathbb{E}_Y[f(X, Y)]^2]}{\mathbb{E}_{X, Y}[f(X, Y)^2]} = \frac{\int_{\Omega} (\int_{\Omega} f(x, y)\pi_x(y)dy)^2 \pi_x(x)dx}{\int_{\Omega} \int_{\Omega} f(x, y)^2 \pi_x(y)\pi_x(x)dydx}.$$

803 Note that the correlation coefficient is only large if it is possible to find some set
 804 of traits which are expected to perform either well or poorly on average, and if these

¹⁴Note that $\rho = 1/2$ guarantees perfect transitivity but $\rho = 0$ does not guarantee that the tournament is perfectly cyclic. A counterexample suffices to explain why. Suppose each competitor chooses rock, paper, or scissors uniformly and independently. Suppose there are three competitors and the tournament is complete. Then, in order for the tournament to be perfectly cyclic, rock must be chosen by one competitor, scissors by another, and paper by the last. There are 6 ways this can happen but 27 possible tournaments, so there is a 21/27 chance the tournament is perfectly transitive. Note that if the network is dense and $\rho = 0$ the network may be predominantly, if not perfectly, cyclic.

805 traits occur sufficiently often. That is, there must be some x such that $|\mathbb{E}_Y[f(x, Y)]|$
 806 is large, and $\pi_x(x)$ is not too small. From this expression, it is not surprising that the
 807 expected strength of transitive competition is monotonically increasing in ρ . If there
 808 is a set of traits x which, on average, either overperform or underperform against
 809 randomly drawn opponents, and are frequently sampled, then a random sample of V
 810 competitors is expected to include some who perform well, and some poorly, against
 811 their neighbors. If, on the other hand, the expected performance conditioned on
 812 traits x is close to neutral, then ρ is small and competition is expected to be cyclic.
 813 In a rock-paper-scissors style game in which competitors are randomly and uniformly
 814 assigned rock, paper, or scissors, conditioning on receiving a particular trait does not
 815 change the probability that an individual with that trait will win most contests, hence
 816 the tournament is expected to be highly cyclic if L is large relative to V .

817 Another way to read (4.18) is as follows. Define the expected performance of
 818 traits x to be $\mathbb{E}_Y[f(x, Y)]$. Then, since $\mathbb{E}_X[\mathbb{E}_Y[f(X, Y)]] = \mathbb{E}_{X,Y}[f(X, Y)] = 0$,
 819 $\mathbb{E}_X[\mathbb{E}_Y[f(X, Y)]^2]$ is the variance in the expected performance given X . Therefore
 820 ρ is the ratio of the variance in the expected performance given X to the variance
 821 in performance. A large variance in the expected performance means we are likely
 822 to sample some competitors who perform well, or poorly, against most opponents.
 823 Consequently, the sampled edge flow is expected to be more transitive than cyclic.

824 Rereading [Theorem 4.1](#) in this way leads to the following insight:

825 **COROLLARY 4.2.** *If the traits W, X, Y are sampled independently from π_x and*
 826 *$F = f(X, Y)$ then the correlation coefficient ρ is proportional to the variance in the*
 827 *expected performance:*

$$828 \quad (4.19) \quad \rho = \frac{1}{\sigma^2} \text{Cov}(f(X, Y), f(X, W)) = \frac{1}{\sigma^2} \text{Var}(\mathbb{E}[F|X]).$$

829 *Let ν be the expected variance in the performance:*

$$830 \quad (4.20) \quad \nu = \frac{1}{\sigma^2} \mathbb{E}[\text{Var}(F|X)].$$

831 *Then $\nu = \frac{1}{\sigma^2} \text{Var}[f(X, Y) - f(X, W)] = 1 - \rho$, so $\mathbb{E}[|F_c|^2]$ is monotonically*
 832 *increasing in ν and $\mathbb{E}[|F_t|^2]$ is monotonically decreasing in ν .*

833 The proof is provided in the supplement and follows from the law of total variance,

$$834 \quad (4.21) \quad \sigma^2 = \text{Var}(F) = \mathbb{E}[\text{Var}(F|X)] + \text{Var}[\mathbb{E}(F|X)] = \sigma^2(\rho + \nu).$$

835 [Theorem 4.1](#) identifies which statistical feature of the trait distribution and perfor-
 836 mance function promotes transitive and suppresses cyclic competition. [Corollary 4.2](#)
 837 identifies which feature suppresses transitive and promotes cyclic competition. Tran-
 838 sitive competition is promoted by uncertainty in expected performance, $\text{Var}[\mathbb{E}(F|X)]$,
 839 and suppressed by expected uncertainty, $\mathbb{E}[\text{Var}(F|X)]$. Conversely, cyclic competition
 840 is suppressed by uncertainty in expected performance, and promoted by expected un-
 841 certainty. If the expected uncertainty in performance is large, then performance is
 842 competitor dependent, hence competition is mostly cyclic.

843 [Theorem 4.1](#) and [Corollary 4.2](#) provide conceptual bridges between uncertainty in
 844 the edge flow, correlation structure on adjacent edges, and network structure (see [Fig-](#)
 845 [ure 9](#)). They establish the intuitive statements that conclude the introduction (p. 4).
 846 For example, the expected uncertainty in the performance of A against a random
 847 competitor is $\sigma^2\nu = \frac{1}{2}\mathbb{E}_X[\text{Var}_Y(f(X, Y)|X)]$. Thus, “*the less predictable the perfor-*
 848 *mance of A against a randomly drawn competitor, the more cyclic the tournament*”.

849 By the equivalence of $\mathbb{E}_X[\text{Var}_Y(f(X, Y)|X)]$ to $\text{Var}(f(X, Y) - f(X, W))$, “the more
850 the performance of A depends on their opponent, the more cyclic the tournament.”

851 **4.3. Generalization.** The trait-performance assumptions are not valid for all
852 tournaments of interest.

853 Nevertheless, the conclusions of the trait-performance can be generalized to sit-
854 uations where the assumptions do not hold. We propose three generalizations. First
855 we consider a situation where performance is only partially determined by traits. Sec-
856 ond, if the network is complete, then the established relationship between expected
857 structure and correlation holds when ρ is replaced with its empirical estimate. The
858 empirical correlation depends only on the observed network, so the relation is an alge-
859 braic fact that is true for all complete networks, whatever the underlying distribution.
860 Third, the trait-performance results hinged on a canonical form for the covariance in
861 the edge flow (4.5). If an edge flow distribution has covariance in the canonical form,
862 then the expected structure of the network satisfies (4.6). Thus, the conclusions re-
863 lating structure to correlation hold for any edge flow distribution with covariance in
864 the canonical form, whether or not that distribution came from a trait-performance
865 model. If we assume a priori that our distribution has a covariance in this form, then
866 ρ is a single parameter that tunes the sampled networks structure.

867 To start, what if performance is influenced by some random factors (such as
868 unmeasured traits) in addition to a set of measured traits? Decompose $\text{Cov}(F)$ using
869 the law of total variation. The first term in the decomposition would be the covariance
870 in the the expected log-odds given the traits, which is a function of randomly drawn
871 traits, so would take the canonical form (4.5) where the performance function $f(x, y)$
872 is replaced with $\mathbb{E}[F|x, y]$. Then, since $\mathbb{E}[|F_t|^2]$ and $\mathbb{E}[|F_c|^2]$ are linear in $\text{Cov}(F)$,
873 the expected sizes of the transitive and cyclic components could each be expressed as a
874 combination of a term contributed by the uncertainty in traits, and a term contributed
875 by the uncertainty in performance given traits. The first term would simplify in the
876 standard way, so the influence of the measured traits on expected network structure
877 would follow as in the trait-performance theorem.

878 Second, we define the empirical correlation $\rho(\mathcal{G}_{\rightleftharpoons})$ and variance $\sigma^2(\mathcal{G}_{\rightleftharpoons})$ associated
879 with a particular competitive network $\mathcal{G}_{\rightleftharpoons}$. The empirical variance and correlation
880 are estimators for the variance and correlation given the observed network. The
881 empirical correlation is the covariance in the edge flow over all pairs of edges sharing
882 an endpoint, divided by the empirical variance in the edge flow. Note that we only
883 have one observation of f per edge, so we need to make some assumption about the
884 expected value of the edge flow. We compute both the covariance and variance under
885 the assumption that the expected edge flow is zero on each edge k . The assumption
886 is valid provided that we would have no way to predict the sign of f_k (whether $i(k)$
887 or $j(k)$ usually wins) from the network topology alone. Then, $\rho(\mathcal{G}_{\rightleftharpoons})$ is the average
888 value of $a_{Ekl}f_{i(k)j(k)}f_{i(l)j(l)}$ over all pairs of edges k, l that share an endpoint, where
889 $a_{Ekl} = 1$ if the edges are consistently oriented, and $a_{Ekl} = -1$ if the edges are
890 inconsistently oriented. The empirical variance $\sigma^2(\mathcal{G}_{\rightleftharpoons})$ is simply $\frac{1}{E}\|f\|^2$.

891 **LEMMA 4.3.** *If the competitive network is complete, has V vertices, E edges, L*
892 *loops, empirical variance $\sigma^2(\mathcal{G}_{\rightleftharpoons})$, and correlation $\rho(\mathcal{G}_{\rightleftharpoons})$ then:*

$$893 \quad (4.22) \quad \frac{1}{E}\|f\|^2 = \sigma^2(\mathcal{G}_{\rightleftharpoons}) \xrightarrow{\text{decompose}} \begin{cases} \frac{1}{E}\|f_t\|^2 = \sigma^2(\mathcal{G}_{\rightleftharpoons}) \left[\frac{(V-1)}{E} + 2\rho(\mathcal{G}_{\rightleftharpoons}) \frac{L}{E} \right] \\ \frac{1}{E}\|f_c\|^2 = \sigma^2(\mathcal{G}_{\rightleftharpoons}) (1 - 2\rho(\mathcal{G}_{\rightleftharpoons})) \frac{L}{E} \end{cases}$$

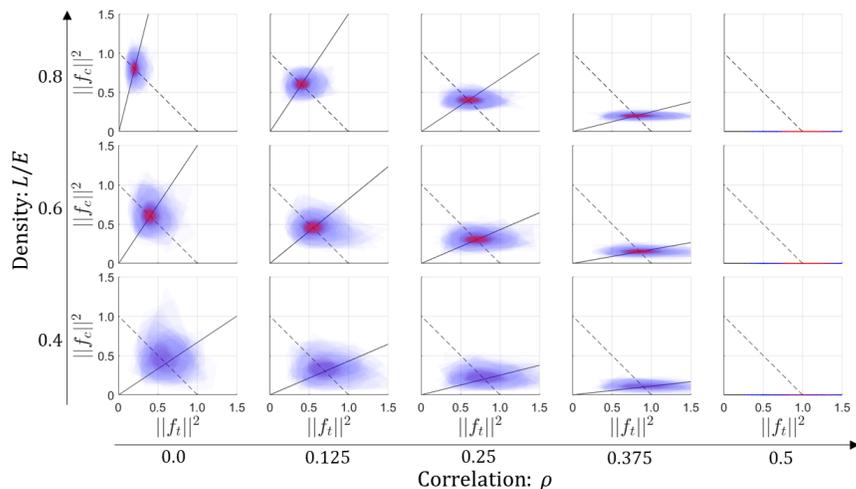


FIG. 10. *Transitivity and intransitivity of sampled networks with varying edge density, number of competitors, and correlation ρ . Each row represents networks with a fixed ratio L/E where L is the number of loops, $E - (V - 1)$, and E is the number of edges. Each column represents a fixed correlation ρ . When $\rho = 0$ the edge flows on all edges are independent. When $\rho = 0.5$ the randomly sampled networks are all perfectly transitive. The blue shaded region is a heat map representing 10^4 sampled networks with 20 competitors. The red shaded region is a heat map representing 10^4 sampled networks with 300 competitors. The topology of each network is sampled randomly from the family of connected Erdos-Renyi networks. The edge flows are sampled from the multivariate Gaussian distribution with mean zero and covariance of form (4.5). The solid black line represents the expected relative sizes of the transitive and intransitive component predicted by equation (4.6). The dashed black line represents the expected total intensity of competition, σ^2 . The intersection of these two lines gives the expected absolute sizes of the transitive and intransitive components. Notice that the trait-performance theorem correctly predicts the relative and absolute sizes of the transitive and intransitive components as a function of L/E , σ , and ρ . Moreover, the more competitors in the network, the tighter the agreement to the expected sizes.*

894 The proof is provided in the supplement.

895 Third, the conclusions of the trait-performance theorem relating correlation and
 896 topology to structure hold as long as the edge flow F has covariance in the canonical
 897 form (4.5). The trait-performance assumptions guarantee that the covariance takes
 898 this form, but an edge flow F may have a covariance in this form whether or not
 899 it is related to an underlying trait-performance model. Thus the conclusions of the
 900 theorem generalize to all edge flow distributions with covariance of the form (4.5).

901 It follows that we can use the trait-performance results to design families of null
 902 models with tunable structure. For example, suppose that we are given a specific
 903 network topology. Then we could sample F from the multivariate Gaussian distri-
 904 bution with mean zero and covariance chosen to match (4.5). By choosing σ^2 and
 905 ρ we uniquely specify the edge flow distribution. Then the expected absolute and
 906 relative sizes of the transitive and cyclic components would be directly controlled by
 907 the choice of σ^2 and ρ . We could tune the overall intensity of competition by varying
 908 σ^2 , and the relative degree of intransitivity by varying ρ . Results from null models of
 909 this kind are demonstrated in Figure 10. The figure demonstrate that it is possible
 910 to define null models with a chosen degree of transitivity by tuning the correlation ρ .

911 Null models of this kind could be useful since many empirical studies involve
 912 complex competition events where reasonable statistical modelling of sampling error

913 is difficult [16, 79]. Absent a statistical error model, the observed edge flow must be
 914 treated as truth, so significance must be computed with respect to a null distribu-
 915 tion. The standard test approximates significance relative to a uniform distribu-
 916 tion of dominance relationships (sign of the edge flow) on a complete network [2, 15, 35].
 917 This significance is only useful so far as the uniform null model is a plausible model
 918 for competition, or as it restricts the space of possible competition structures. The
 919 fact that most studies identify significant transitivity suggests that the uniform dis-
 920 tribution is rarely plausible. Failure to match a uniform distribution also does not
 921 limit the competitive structure significantly, since, as demonstrated above, it is easy
 922 to construct null models that produce intermediate levels of transitivity.

923 In fact, complete networks with edge flow drawn uniformly are the *most* cyclic
 924 edge flow distribution with covariance of the form (4.5) since they are simultaneously
 925 as dense and uncorrelated as possible. Complete networks with uniform i.i.d. edge
 926 flow live in the upper left-hand corner of Figure 10. It is not surprising that most
 927 empirical networks are more transitive than the most cyclic ensemble. For this reason,
 928 significance computed against the uniform complete null model may depend primarily
 929 on the number of imputed edges, as observed in [63, 37, 22], rather than true structure.

930 The family of null models proposed here could generalize the standard random-
 931 ization test in two useful ways. First, it allows for arbitrary network topology, so
 932 does not require imputing missing edges which reduces the strength of the test [63].
 933 Second, the expected degree of transitivity in the null model can be tuned using one
 934 parameter, ρ . Once ρ is chosen, we could compute the probability of observing a
 935 network that is more or less transitive or intransitive relative to random networks
 936 with correlation ρ . Thus significance could be measured against a flexible range of
 937 networks with varying degrees of transitivity. Then it would be possible to search
 938 over $\rho \in [0, 0.5]$ to find the largest and smallest ρ which produce random networks
 939 with significantly different structure than the observed network. The interval between
 940 these upper and lower bounds on ρ would define an interval in each transitivity mea-
 941 sure that could plausibly correspond to the observed network. Thus, expanding the
 942 family of null models would allow more flexible, informative, significance testing, as
 943 well as interval estimation of the measures of competitive structure.

944 **5. Discussion.** The discrete HHD provides a natural, unified method for rank-
 945 ing and measuring intransitivity via a decomposition into perfectly transitive and
 946 cyclic components. The expected size of these components can be computed from
 947 the correlation structure of the edge flow. Using a trait-performance model simplifies
 948 this structure. We provide an illustrative example in the supplement. Note that the
 949 trait-performance conclusions are valid whenever the assumptions hold, whether or
 950 not the relevant traits or performance function are known. Thus the assumptions can
 951 be tested by checking whether the observed correlation structure matches (4.5).

952 Further theoretical work could address random network topologies. If the network
 953 is sampled independently of the edge flow then the results of Theorem 4.1 are largely
 954 unchanged, so one might consider random networks whose topology is coupled to the
 955 competitor traits. For example, neighbors in the network might have correlated traits.
 956 Future work could also investigate null models with different covariance structures.

957 We emphasize that the HHD can be applied to analyze a tournament independent
 958 of a null model. Code for implementing our methods are available on [github](#). In
 959 particular, our methods can be extended to analyze data from real tournaments. By
 960 studying win-loss records it is possible to infer the log odds, and thus estimate the
 961 components of the HHD. The estimation problem is saved for future work.

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